

CHAOS FOR COWEN-DOUGLAS OPERATORS

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ABSTRACT. In this article, we provide a sufficient condition which gives Devaney chaos and distributional chaos for Cowen-Douglas operators. In fact, we obtain a distributionally chaotic criterion for bounded linear operators on Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A discrete dynamical system is simply a continuous mapping $f : X \rightarrow X$ where X is a complete separable metric space. For $x \in X$, the orbit of x under f is $Orb(f, x) = \{x, f(x), f^2(x), \dots\}$ where $f^n = f \circ f \circ \dots \circ f$ is the n^{th} iterate of f obtained by composing f with n times.

Recall that f is transitive if for any two non-empty open sets U, V in X , there exists an integer $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. It is well known that, in a complete metric space without isolated points, transitivity is equivalent to the existence of dense orbit ([15]). f is weakly mixing if $(f \times f, X \times X)$ is transitive. f is strongly mixing if for any two non-empty open sets U, V in X , there exists an integer $m \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq m$. f has sensitive dependence on initial conditions (or simply f is sensitive) if there is a constant $\delta > 0$ such that for any $x \in X$ and any neighborhood U of x , there exists a point $y \in U$ such that $d(f^n(x), f^n(y)) > \delta$, where d denotes the metric on X .

In 1975, Li and Yorke [9] observed complicated dynamical behavior for the class of interval maps with period 3. This phenomena is currently known under the name of Li-Yorke chaos. Therefrom, several kinds of chaos were well studied. In the present article, we focus on Devaney chaos and distributional chaos.

Following Devaney [3],

Definition 1.1. Let (X, f) be a dynamical system. f is chaotic if

- (D1) f is transitive;
- (D2) the periodic points for f are dense in X ; and
- (D3) f has sensitive dependence on initial conditions.

It was shown by Banks et. al. ([1]) that $(D1) + (D2)$ implies $(D3)$ for any aperiodic system. Devaney chaos is a stronger version chaos than Li-Yorke chaos, which is given by Huang and Ye [8] and Mai [12].

From Schweizer and Smítal's paper [14], distributional chaos is defined in the following way.

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For any pair $\{x, y\} \subset X$ and any $n \in \mathbb{N}$, define distributional function $F_{xy}^n : \mathbb{R} \rightarrow [0, 1]$:

$$F_{xy}^n(\tau) = \frac{1}{n} \# \{0 \leq i \leq n : d(f^i(x), f^i(y)) < \tau\}.$$

Furthermore, define

$$\begin{aligned} F_{xy}(\tau) &= \liminf_{n \rightarrow \infty} F_{xy}^n(\tau), \\ F_{xy}^*(\tau) &= \limsup_{n \rightarrow \infty} F_{xy}^n(\tau) \end{aligned}$$

Both F_{xy} and F_{xy}^* are nondecreasing functions and may be viewed as cumulative probability distributional functions satisfying $F_{xy}(\tau) = F_{xy}^*(\tau) = 0$ for $\tau < 0$.

Definition 1.2. $\{x, y\} \subset X$ is said to be a distributionally chaotic pair, if

$$F_{xy}^*(\tau) \equiv 1, \quad \forall \tau > 0 \quad \text{and} \quad F_{xy}(\epsilon) = 0, \quad \exists \epsilon > 0.$$

Furthermore, f is called distributionally chaotic, if there exists an uncountable subset $D \subseteq X$ such that each pair of two distinct points is a distributionally chaotic pair. Moreover, D is called a distributionally ϵ -scrambled set.

Distributional chaos always implies Li-Yorke chaos, as it requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic. The converse implication is not true in general. However in practice, even in the simple case of Li-Yorke chaos, it might be quite difficult to prove chaotic behavior from the very definition. Such attempts have been made in the context of linear operators (see [4, 5]). Further results of [4] were extended in [13] to distributional chaos for the annihilation operator of a quantum harmonic oscillator. More about distributional chaos, one can see [16, 17, 10, 11, 18].

Let \mathcal{H} be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on \mathcal{H} . We are interesting in a family of operators given by Cowen and Douglas [2].

Definition 1.3. For Ω a connected open subset of \mathbb{C} and n a positive integer, let $\mathcal{B}_n(\Omega)$ denotes the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy:

- (a) $\Omega \subseteq \sigma(T) = \{\omega \in \mathbb{C} : T - \omega \text{ not invertible}\}$;
- (b) $\text{ran}(T - \omega) = \mathcal{H}$ for ω in Ω ;
- (c) $\bigvee_{\omega \in \Omega} \ker(T - \omega) = \mathcal{H}$; and
- (d) $\dim \ker(T - \omega) = n$ for ω in Ω .

We have known some properties of these operators from [2].

Proposition 1.4. Let $T \in \mathcal{B}_n(\Omega)$ and $\omega_0 \in \Omega$. Then $\bigvee_{k=1}^{\infty} \ker(T - \omega_0)^k = \mathcal{H}$.

Proposition 1.5. If $\Omega_0 \subseteq \Omega$ is a bounded connected open subset of \mathbb{C} , then $B_n(\Omega) \subseteq B_n(\Omega_0)$.

In this paper, S is always used to denote the unit circle in \mathbb{C} . In the next section, we provide a sufficient condition $\Omega \cap S \neq \emptyset$ which gives Devaney chaos for Cowen-Douglas operators. In the last section, we obtain a distributionally chaotic criterion for bounded linear operators on Banach spaces. Applied by this distributionally chaotic criterion, $\Omega \cap S \neq \emptyset$ is also a sufficient condition which gives distributional chaos for Cowen-Douglas operators.

2. DEVANEY CHAOS FOR COWEN-DOUGLAS OPERATORS

Proposition 2.1. *Let $T \in \mathcal{B}_n(\Omega)$. If $\Omega \cap S \neq \phi$, then T is strongly mixing.*

Proof. Suppose U and V be arbitrary open subsets in \mathcal{H} . We have $\epsilon > 0$ and open subsets U' and V' such that

$$B(u', \epsilon) \subseteq U \quad \text{and} \quad B(v', \epsilon) \subseteq V,$$

for any $u' \in U'$ and any $v' \in V'$. Since Ω is a connected open subset and $\Omega \cap S \neq \phi$, there are two bounded connected open subsets Ω_1 and Ω_2 in Ω such that

$$\sup_{\alpha \in \Omega_1} |\alpha| = \lambda < 1 \quad \text{and} \quad \inf_{\beta \in \Omega_2} |\beta| = \rho > 1.$$

By proposition 1.5, there exist two points $x \in U'$ and $y \in V'$ with the following forms:

$$x = \sum_{i=1}^t x_i \quad \text{and} \quad y = \sum_{j=1}^l y_j,$$

where $x_i \in \ker(T - \lambda_i)$, $\lambda_i \in \Omega_1$ and $y_j \in \ker(T - \rho_j)$, $\rho_j \in \Omega_2$.

Now let $M = \max\{\sum_{i=1}^t \|x_i\|, \sum_{j=1}^l \|y_j\|\}$. Then there is a positive integer N such that for each $k \geq N$,

$$\lambda^k < \epsilon/M \quad \text{and} \quad \rho^{-k} < \epsilon/M.$$

Given any $k \geq N$, let $u(k) = x + \sum_{j=1}^l \rho_j^{-k} y_j$. Obviously,

$$\|u(k) - x\| = \left\| \sum_{j=1}^l \rho_j^{-k} y_j \right\| \leq \sum_{j=1}^l |\rho_j^{-k}| \cdot \|y_j\| \leq \rho^{-k} \sum_{j=1}^l \|y_j\| < \epsilon,$$

so $u(k) \in U$. On the other hand,

$$\|T^k u(k) - y\| = \left\| \sum_{i=1}^t \lambda_i^k x_i \right\| \leq \sum_{i=1}^t |\lambda_i^k| \cdot \|x_i\| \leq \lambda^k \sum_{i=1}^t \|x_i\| < \epsilon,$$

that implies $T^k u(k) \in V$. Hence $T^k(U) \cap V \neq \phi$ and consequently T is strongly mixing. □

Proposition 2.2. *Let $T \in \mathcal{B}_n(\Omega)$. If $\Omega \cap S \neq \phi$, then $\text{Per}(T)$ is dense in \mathcal{H} .*

Proof. Let $\Delta = \{e^{2\pi r i} : \text{for all rational numbers } r\}$. Then Δ is dense in S , and one can see that for each $\delta \in \Delta$, there exists a positive integer $m(\delta)$ such that δ is a root of the equation $z^{m(\delta)} = 1$. Since Ω is a connected open subset and $\Omega \cap S \neq \phi$, we have $\Omega \cap \Delta \neq \phi$. Now let $s \in \Omega \cap \Delta$. If $x \in \ker(T - s)^k$ for any k , then $T^{km(s)}(x) = x$ and hence $\bigcup_{k=1}^{\infty} \ker(T - s)^k \subseteq \text{Per}(T)$. Therefore, by proposition 1.4, $\text{Per}(T)$ is dense in \mathcal{H} . □

By Proposition 2.1 and 2.2, one can see the following result immediately.

Theorem 2.3. *Let $T \in \mathcal{B}_n(\Omega)$. If $\Omega \cap S \neq \phi$, then T is Devaney chaotic.*

Remark 2.4. Notice that $\Omega \cap S \neq \emptyset$ is not a necessary condition for Devaney chaos for Cowen-Douglas operators. As well-known, the backward shift operator T , with the weight sequence $\{\omega_n = \frac{n+1}{n}\}_{n=1}^\infty$, is a Devaney chaotic Cowen-Douglas operator. However, the largest connected open domain Ω for T , which admits $T \in B_1(\Omega)$, is the unit open disk and hence is disjoint with S .

3. DISTRIBUTIONALLY CHAOTIC CRITERION AND ITS APPLICATION ON COWEN-DOUGLAS OPERATORS

First of all, we'll give a new concept which is very useful to prove an bounded linear operator is distributional chaotic.

Definition 3.1. Let X be a Banach space and let $T \in \mathcal{L}(X)$. T is called norm-unimodal, if we have a constant $\gamma > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X$ satisfying

$$(NU1) \quad \lim_{k \rightarrow \infty} \|T^k x_m\| = 0,$$

$$(NU2) \quad \|T^i x_m\| \geq \gamma^i \|x_m\|, \quad i = 1, 2, \dots, m.$$

Furthermore, such γ is said to be a norm-unimodal constant for the norm-unimodal operator T .

Remark 3.2. If x is the point referred to in the above definition, then for any $c \in \mathbb{C}$, cx has the same properties as x because of the linearity of T . Therefore, we can select a point with arbitrary non-zero norm satisfying the same conditions.

Theorem 3.3 (Distributionally Chaotic Criterion). *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If T is norm-unimodal, then T is distributionally chaotic.*

Proof. Let $R = \|T\|$ and let γ be a norm-unimodal constant for T . Suppose $\{\epsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers decreasing to zero. First of all, fix $N_1 \in \mathbb{N}$ (for example, set $N_1 = 2$). Then there is x_1 such that $\|x_1\| = 1$ and

$$\lim_{k \rightarrow \infty} \|T^k x_1\| = 0, \quad \text{and} \quad \|T^i x_1\| \geq \gamma^i \|x_1\|, \quad i = 1, \dots, N_1.$$

So we can choose M_1 such that $\|T^n x\| < \epsilon_1$ for any $n \geq M_1$. For convenience, let $N'_1 = 0$. Then $\|T^i x_1\| \geq 1$, $i = N'_1, \dots, N_1$.

Now we'll construct a sequence of points $\{x_k\}_{k=1}^\infty$ associated with three sequences of integers $\{N_k\}_{k=1}^\infty$, $\{N'_k\}_{k=1}^\infty$ and $\{M_k\}_{k=1}^\infty$ such that for every $k \geq 2$

$$(I) \quad \|x_k\| = R^{-M_{k-1}} \cdot 2^{-k} \cdot \epsilon_{k-1};$$

$$(II) \quad \|T^i x_k\| \geq \gamma^i \|x_k\|, \quad i = 1, \dots, N_k;$$

$$(III) \quad \gamma^{N'_k} \cdot R^{-M_{k-1}} \cdot 2^{-k} \cdot \epsilon_{k-1} > 1;$$

$$(IV) \quad \frac{N_k - N'_k}{N_k} > \frac{k-1}{k};$$

$$(V) \quad \sum_{j=1}^k \|T^n x_j\| < \epsilon_k, \quad \text{for any } n \geq M_k.$$

Select $N'_2 \in \mathbb{N}$ with $\gamma^{N'_2} \cdot R^{-M_1} \cdot 2^{-2} \cdot \epsilon_1 > 1$. Consequently, we have $N_2 \in \mathbb{N}$ such that $\frac{N_2 - N'_2}{N_2} > \frac{2-1}{2} = \frac{1}{2}$. And then there is x_2 such that $\|x_2\| = R^{-M_1} \cdot 2^{-2} \cdot \epsilon_1$ and

$$\lim_{k \rightarrow \infty} \|T^k x_2\| = 0, \quad \text{and} \quad \|T^i x_2\| \geq \gamma^i \|x_2\|, \quad i = 1, \dots, N_2.$$

So we can choose M_2 such that $\|T^n x_1\| + \|T^n x_2\| < \epsilon_2$ for any $n \geq M_2$.

Continue in this manner. If we have obtained $\{x_k\}_{k=1}^m$, $\{N_k\}_{k=1}^m$, $\{N'_k\}_{k=1}^m$ and $\{M_k\}_{k=1}^m$ such that for each $k = 2, \dots, m$

$$(1) \quad \|x_k\| = R^{-M_{k-1}} \cdot 2^{-k} \cdot \epsilon_{k-1};$$

- (2) $\|T^i x_k\| \geq \gamma^i \|x_k\|$, $i = 1, \dots, N_k$;
- (3) $\gamma^{N'_k} \cdot R^{-M_{k-1}} \cdot 2^{-k} \cdot \epsilon_{k-1} > 1$;
- (4) $\frac{N_k - N'_k}{N_k} > \frac{k-1}{k}$;
- (5) $\sum_{j=1}^k \|T^n x_j\| < \epsilon_k$, for any $n \geq M_k$;

Select $N'_{m+1} \in \mathbb{N}$ with $\gamma^{N'_{m+1}} \cdot R^{-M_m} \cdot 2^{-(m+1)} \cdot \epsilon_m > 1$. Consequently, we have $N_{m+1} \in \mathbb{N}$ such that $\frac{N_{m+1} - N'_{m+1}}{N_{m+1}} > \frac{m+1-1}{m+1} = \frac{m}{m+1}$. And then there is x_{m+1} such that $\|x_{m+1}\| = R^{-M_m} \cdot 2^{-(m+1)} \cdot \epsilon_m$ and

$$\lim_{k \rightarrow \infty} \|T^k x_{m+1}\| = 0, \text{ and } \|T^i x_{m+1}\| \geq \gamma^i \|x_{m+1}\|, \quad i = 1, \dots, N_{m+1}.$$

So we can choose M_{m+1} such that $\sum_{j=1}^{m+1} \|T^n x_j\| < \epsilon_{m+1}$ for any $n \geq M_{m+1}$.

Therefore, we obtain a sequence of points $\{x_k\}_{k=1}^\infty$ associated with three sequences of integers $\{N_k\}_{k=1}^\infty$, $\{N'_k\}_{k=1}^\infty$ and $\{M_k\}_{k=1}^\infty$ satisfying conditions (I-V). Moreover, conditions (I-III) imply following statement:

(VI) $\sum_{k=1}^\infty \|x_k\|$ is finite.

(VII) For each p , $\|T^i x_k\| < 2^{-k} \epsilon_{k-1}$, for any $k > p$ and any $1 \leq i \leq M_p$. Hence,
 $\sum_{k=p+1}^\infty \|T^i x_k\| < \sum_{k=p+1}^\infty 2^{-k} \epsilon_{k-1} < \epsilon_p$, for any $1 \leq i \leq M_p$.

(VIII) For each k , $\|T^i x_k\| \geq 1$, $i = N'_k, \dots, N_k$.

Notice $M_k > N_k > N'_k > M_{k-1}$ for each k by the manner of our construction. Then we have

(V') $\sum_{j=1}^{k-1} \|T^n x_j\| < \epsilon_{k-1}$, for $n = N'_k, \dots, N_k$.

(VII') For each p , $\sum_{k=p+1}^\infty \|T^n x_k\| < \epsilon_p$, $n = N'_p, \dots, N_p$.

Let $\Sigma_2 = \{0, 1\}^\mathbb{N}$ be a symbolic space with two symbols. According to condition (VI), we can define a map $f : \Sigma_2 \rightarrow X$ as follows,

$$f(\xi) = \sum_{k=1}^\infty \xi_k x_k,$$

for every element $\xi = (\xi_1, \xi_2, \dots) \in \Sigma_2$.

Obviously one can get an uncountable subset $D \in \Sigma_2$ such that for any two distinct $\xi, \xi' \in D$, ξ and ξ' have infinite coordinates different and infinite coordinates equivalent. Then

$$d(f(\xi), f(\xi')) = \|f(\xi) - f(\xi')\| = \left\| \sum_{k=1}^\infty (\xi_k - \xi'_k) x_k \right\|.$$

Set $\theta = (\theta_1, \theta_2, \dots) = (\xi_1 - \xi'_1, \xi_2 - \xi'_2, \dots)$. Then $d(f(\xi), f(\xi')) = \left\| \sum_{k=1}^\infty \theta_k x_k \right\|$.

Note that the possible values of $\xi_k - \xi'_k$ are only 0, -1 or 1, and θ has infinite coordinates being zero and infinite coordinates being nonzero.

Now we'll prove that $\{f(\xi), f(\xi')\}$ is a distributionally chaotic pair.

Let $z = \sum_{k=1}^{\infty} \theta_k x_k$. Suppose $\{k_q\}_{q=1}^{\infty}$ be the infinite subsequence such that the $k_q - th$ coordinate of θ is nonzero (1 or -1) and $\{k_r\}_{r=1}^{\infty}$ be the infinite subsequence such that the $k_r - th$ coordinate of θ is zero.

By (V'), (VII') and (VIII), for $n = N'_{k_q}, \dots, N_{k_q}$

$$\|T^n z\| \geq \|T^n(\theta_{k_q} x_{k_q})\| - \sum_{j=1}^{k_q-1} \|T^n x_j\| - \sum_{j=k_q+1}^{\infty} \|T^n x_j\| > 1 - \epsilon_{k_q-1} - \epsilon_{k_q}.$$

Since $\{\epsilon_k\}_{k=1}^{\infty}$ decrease to zero, then

$$\begin{aligned} F_{f(\xi)f(\xi')}(\frac{1}{2}) &= \liminf_{n \rightarrow \infty} F_{f(\xi)f(\xi')}^n(\frac{1}{2}) \\ &\leq \liminf_{q \rightarrow \infty} F_{f(\xi)f(\xi')}^{N_{k_q}}(\frac{1}{2}) \\ &\leq \lim_{q \rightarrow \infty} \frac{N'_{k_q}}{N_{k_q}} \\ &\leq \lim_{q \rightarrow \infty} \frac{1}{k_q} = 0. \end{aligned}$$

On the other hand, for $n = N'_{k_r}, \dots, N_{k_r}$

$$\|T^n z\| \leq \|T^n(\theta_{k_r} x_{k_r})\| + \sum_{j=1}^{k_r-1} \|T^n x_j\| + \sum_{j=k_r+1}^{\infty} \|T^n x_j\| \leq \epsilon_{k_r-1} + \epsilon_{k_r}.$$

Since $\{\epsilon_k\}_{k=1}^{\infty}$ decrease to zero, then for any $\tau > 0$

$$\begin{aligned} F_{f(\xi)f(\xi')}^*(\tau) &= \limsup_{n \rightarrow \infty} F_{f(\xi)f(\xi')}^n(\tau) \\ &\geq \limsup_{r \rightarrow \infty} F_{f(\xi)f(\xi')}^{N_{k_r}}(\tau) \\ &\geq \lim_{r \rightarrow \infty} \frac{N_{k_r} - N'_{k_r} + 1}{N_{k_r}} \\ &\geq \lim_{r \rightarrow \infty} \frac{k_r - 1}{k_r} = 1. \end{aligned}$$

Therefore, $\{f(\xi), f(\xi')\}$ is a distributionally chaotic pair for any distinct points $\xi, \xi' \in D$ and hence $f(D)$ is a distributionally ϵ -scrambled set. This ends the proof. \square

In the present article, it has been filled by this conclusion. In fact, we could extend it in some way. For instance, we have

Theorem 3.4 (Weakly Distributionally Chaotic Criterion). *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If for any sequence of positive numbers C_m increasing to $+\infty$, there exist $\{x_m\}_{m=1}^{\infty}$ in X satisfying*

$$(WNU1) \quad \lim_{k \rightarrow \infty} \|T^k x_m\| = 0.$$

(WNU2) There is a sequence of positive integers N_m increasing to $+\infty$, such that $\lim_{m \rightarrow \infty} \frac{\#\{0 \leq i \leq N_m; \|T^i x_m\| \geq C_m \|x_m\|\}}{N_m} = 1$.

Then T is distributionally chaotic.

The proof is similar to Theorem 3.3. According to Grosse-Erdmann's characterization [7], it is not difficult to see that Devaney chaotic backward shift operators satisfy the conditions of Theorem 3.4. So one can get the following conclusion immediately.

Corollary 3.5. *If T is a Devaney chaotic backward shift operator, then T is distributionally chaotic.*

Remark 3.6. This result is contained in a recent article [6], in which F. Martínez-Giménez, P. Oprocha and A. Peris, considered distributional chaos for shift operators.

Applying this Distributionally Chaotic Criterion 3.3, we'll provide a sufficient condition which gives distributional chaotics for Cowen-Douglas operators.

Theorem 3.7. *Let $T \in \mathcal{B}_n(\Omega)$. If $\Omega \cap S \neq \emptyset$, then T is norm-unimodal. Consequently, T is distributionally chaotic.*

Proof. Since Ω is a connected open subset and $\Omega \cap S \neq \emptyset$, there exists $\beta \in \Omega$ with $|\beta| > 1$. Furthermore, we can select a nontrivial $y \in \ker(T - \beta)$. Let $1 < \gamma < |\beta|$ be a constant. Given any $m \in \mathbb{N}$, set $\epsilon < \frac{\|y\|}{2} \cdot \min\{1, \frac{|\beta|^i - \gamma^i}{|\beta|^i + 1}, 1 \leq i \leq m\}$. Then $U = \bigcap_{i=0}^m T^{-i}(B(T^i y, \epsilon))$ is an open neighborhood of y . Then for any $z \in U$,

$$\|T^i z\| \geq \|T^i y\| - \epsilon = |\beta|^i \|y\| - \epsilon \geq |\beta|^i \|z\| - (|\beta|^i + 1)\epsilon \geq \gamma^i \|z\|, \quad i = 1, \dots, m$$

By hypothesis, one can obtain a bounded connected open subset Ω_1 such that

$$\sup_{\alpha \in \Omega_1} |\alpha| = \lambda < 1.$$

By proposition 1.5, there exists a point $x \in U$ satisfying

$$x = \sum_{j=1}^t x_j,$$

where $x_j \in \ker(T - \lambda_j)$, $\lambda_j \in \Omega_1, j = 1, \dots, t$. Then this x is the point we hope to get. One hand,

$$\lim_{k \rightarrow \infty} \|T^k x\| = \lim_{k \rightarrow \infty} \left\| \sum_{j=1}^t \lambda_j^k x_j \right\| \leq \lim_{k \rightarrow \infty} \sum_{j=1}^t |\lambda_j^k| \cdot \|x_j\| \leq \left(\sum_{j=1}^t \|x_j\| \right) \lim_{k \rightarrow \infty} \lambda^k = 0.$$

On the other hand, according to the previous statement and $x \in U$, we have

$$\|T^i x\| \geq \gamma^i \|x\|, \quad i = 1, 2, \dots, m.$$

Therefore T is norm-unimodal and hence T is distributionally chaotic by Theorem 3.3. \square

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